Expanding brackets

We often need to expand (multiply out) brackets in order to simplify an expression. Various methods may be employed, many people use the FOIL method (shown below), but the key point to remember is that everything on the *inside* needs to be multiplied by everything on the *outside*.

- $x(2x+3y^2)=2x^2+3xy^2$

- Expand $(x+2)(x-3)$ using the FOIL method (First, Outside, Inside, Last)

$$(x+2)(x-3)=x^2-3x+2x-6=x^2-x-6$$

With care, we can expand brackets containing any number of terms,

- $$\begin{aligned}(3a-b-c)(a+2b+3c)&=3a(a+2b+3c)-b(a+2b+3c)-c(a+2b+3c)\\&=3a^2+6ab+9ac-ab-2b^2-3bc-ac-2bc-3c^2\\&=3a^2-2b^2-3c^2+5ab-5bc+8ac\end{aligned}$$

Certain results are important and it is worth the effort to learn them…

- $(a+b)^2=a^2+ab+ab+b^2=a^2+2ab+b^2$

- $(a-b)^2=a^2-ab-ab+b^2=a^2-2ab+b^2$

- $(a+b)(a-b)=a^2-ab+ab-b^2=a^2-b^2$

Therefore,

- $(3x-2y)^2=(3x)^2-2(3x)(2y)+(2y)^2=9x^2-12xy+4y^2$

- $(3x-2y)(3x+2y)=(3x)^2-(2y)^2=9x^2-4y^2$

Factorising expressions

Whilst it is important to be able to expanded brackets, it is possibly more-important to be able to reverse the process; that is, to be able to factorise an expression.

Common factors

Some expressions can be factorised by identifying common factors.

- $3xy - 12x^2 = 3(xy - 4x^2) = 3x(y - 4x)$ *this expression has two common factors*

Four-term expressions

Some expressions can be factorised by grouping in pairs.

- $2ax + 3ay - 4bx - 6by = a(2x + 3y) - 2b(2x + 3y) = (a - 2b)(2x + 3y)$

$(2x + 3y)$ is now a common factor

Quadratics

Depending on the particular quadratic, the process of factorisation may be easy or difficult.

Using the difference of two squares

Be on the look out for these situations,

- $x^2 - 4y^2 = (x)^2 - (2y)^2 = (x + 2y)(x - 2y)$
- $8x^2 - 50 = 2(4x^2 - 25) = 2(2x + 5)(2x - 5)$

When the coefficient of x^2 is one

Simply find two numbers that *multiply* to give the constant and *sum* to give the coefficient of x

- $x^2 + x - 6 = (x + 3)(x - 2)$ *multiply to give -6 and add to give $+1$; i.e. 3 and -2*

When the coefficient of x^2 is *not* one

This is more-difficult. For example, if we needed to factorise $4x^2 - 4x - 15$, the solution could be of the form $(4x + ?)(x + ??)$ or $(2x + ?)(2x + ??)$. If you are lucky you might be able to spot the correct factorisation, but most people would have to resort to the following algorithm.

1.	Multiply the coefficient of x^2 by the constant term	$4 \times -15 = -60$
2.	Find factors of -60 that sum to give the coefficient of x (i.e. -4)	$+6 - 10 = -4$
3.	Split the middle term using these numbers	$4x^2 + 6x - 10x - 15$
4.	Factorise the first two terms and then the last two terms	$2x(2x + 3) - 5(2x + 3)$
5.	Complete the factorisation... easy!	$(2x - 5)(2x + 3)$

Quadratic equations

The ability to calculate the roots of a quadratic equation is extremely useful. Quadratic equations occur in the most unlikely areas of mathematics – the flight of a projectile, for example.

Please note that the problem may require you to rearrange an equation into the form $ax^2 + bx + c = 0$ *before* attempting to solve it.

$$x - 67 + 3x^2 = 9 + 6x - 2x^2$$
$$5x^2 - 5x - 76 = 0$$

Factorisation

We can use factorisation (see above for details) to solve $ax^2 + bx + c = 0$.

- Solve $x^2 + x - 6 = 0$
  - As we have seen above, $x^2 + x - 6 = 0$
    $(x+3)(x-2) = 0$
  - Now if the left-hand side is equal to zero, either $(x + 3) = 0$ or $(x - 2) = 0$
  - Therefore, the **roots** of the equations are $x = -3$ and $x = 2$.
  - The **solution** to the equation is the set $\{-3,2\}$; i.e. *all* roots to the equation.
- Solve $4x^2 - 4x - 15 = 0$
  - As we have seen above, $4x^2 - 4x - 15 = 0$
    $(2x-5)(2x+3) = 0$
  - Employing the same logic as before we see that the roots are $x = 2\frac{1}{2}$ and $x = -1\frac{1}{2}$

Assessment questions

Expanding brackets

Expand and simplify the following. **NB *no* answers given.**

1. $(x+2)(x-5)$	2. $(3x-2y)(2x+y)$	3. $(2x+1)(2x-1)$	4. $(a+3b)^2$
5. $(4x-7)(2x+3)$	6. $(4x-7)(3x-1)$	7. $(a-b+c)(2a+b-2c)$	8. $(5x-9)^2$
9. $(x^2+7)(x^2-7)$	10. $(2x^2+x+2)(x^2-3x-4)$		

Factorising expressions

Factorise the following expression. **NB *no* answers given.**

1. x^2+6x+5	2. $x^2-8x-20$	3. $t^2+5t-36$	4. $ad+bd-3ac-3bc$
5. $2x^2+11x+15$	6. $5x^2-17x+6$	7. $3x^2-7x-6$	8. y^2-64
9. $15+x-2x^2$	10. $5x^2-125$	11. $9x^2-12xy+4y^2$	12. $(x+1)^2-y^2$
13. x^4-8x^2-9	14. $6x^2-19x+10$	15. $p^2-q^2-5p+5q$	

Completing the square

The process of completing the square involves re-writing ($ax^2 + bx + c$) as $a(x+p)^2 + q$; that is, a 'square' plus an 'adjustment'.

The case when a = 1

For example, let's put the quadratic equation $x^2 - 4x - 3 = 0$ into completed square form.

- Clearly, if we wish to end up with $x^2 - 4x$, we need to begin with $(x-2)^2$
- $(x-2)^2 = x^2 - 4x + 4$, which is nearly the quadratic required
- However, we don't want '+ 4', we want '– 3' and so we must subtract 7 – this is the 'adjustment'.

$$x^2 - 4x - 3 = (x-2)^2 - 7$$

> If the coefficient of x^2 is *one* then the number in the bracket is half of the coefficient of x

- Complete the square for $x^2 + 6x + 1$

$$x^2 + 6x + 1 = 0$$
$$(x+3)^2 - 9 + 1 = 0$$
$$(x+3)^2 - 8 = 0$$

The case where a ≠ 1

In the case where a ≠ 1, we start by taking *a* out as a factor; then we complete the square for the quadratic inside the bracket; before finally multiplying out.

- Complete the square for $2x^2 - x + 1$

The first step is take out the coefficient of x^2 as a factor

$$2(x^2 - \frac{1}{2}x + \frac{1}{2})$$

Now we complete the square as before…

> As before, this number is half the coefficient of x ; i.e. half of $-\frac{1}{2}$

$$2\left(x^2 - \frac{1}{2}x + \frac{1}{2}\right) = 2\left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} + \frac{1}{2}\right]$$

> Rather than trying to work out the 'adjustment' in one go, simply subtract the constant from the square.

$$= 2\left[\left(x - \frac{1}{4}\right)^2 + \frac{7}{16}\right]$$

> Finally, multiply out to leave the quadratic in completed square form.

$$= 2\left(x - \frac{1}{4}\right)^2 + \frac{7}{8}$$

Indices

Definitions

In a^m, a is the base and m is the index. Please note that the plural of index is indices, not indic*ies*.

Index laws

If two quantities are in the same base then the following rules apply:

$$a^m \times a^n = a^{(m+n)}$$

$$a^m \div a^n = \frac{a^m}{a^n} = a^{(m-n)}$$

$$(a^m)^n = a^{mn}$$

$$a^0 = 0$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Do not confuse these two rules...

Questions may require you to convert all quantities to the same base and/or combine several of the rules above.

- Find the value of

(a) $81^{\frac{1}{2}}$,

$$81^{\frac{1}{2}} = \sqrt{81} = 9$$

(b) $81^{\frac{3}{4}}$,

$$81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3 = (3)^3 = 27$$

(c) $81^{-\frac{3}{4}}$.

$$81^{-\frac{3}{4}} = \frac{1}{81^{\frac{3}{4}}} = \frac{1}{\left(\sqrt[4]{81}\right)^3} = \frac{1}{(3)^3} = \frac{1}{27}$$

- Evaluate $16^{-\frac{3}{4}}$

$$16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}}$$

$$= \frac{1}{\sqrt[4]{16^3}} = \frac{1}{(\sqrt[4]{16})^3}$$

$$= \frac{1}{2^3} = \frac{1}{8}$$

We can either cube 16 and *then* find the fourth-root; or we can find the fourth-root of 16 and cube the answer.

Obviously, one option is much easier than the other.

Surds

A surd is an *irrational* number. Often it includes the positive root of a non-square number; for example, $\sqrt{3}$ and ($2-\sqrt{5}$) are surds, but $\sqrt{4}$ is not a surd, as $\sqrt{4}=2$.

Surd laws

$$\sqrt{ab}=\sqrt{a}\sqrt{b} \qquad \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$$

When simplifying surds it is important to try and identify *square* factors.

Examples:

- Simplify the following

(a) $\sqrt{72}$

Square number

$$\begin{aligned}\sqrt{72}&=\sqrt{36}\times\sqrt{2}\\&=6\sqrt{2}\end{aligned}$$

(b) $\sqrt{27}+4\sqrt{12}$

$$\begin{aligned}\sqrt{27}+4\sqrt{12}&=\sqrt{9}\sqrt{3}+4(\sqrt{4}\sqrt{3})\\&=3\sqrt{3}+4(2\sqrt{3})\\&=3\sqrt{3}+8\sqrt{3}\\&=11\sqrt{3}\ .\end{aligned}$$

(c) $(4+\sqrt{3})^2-2(1-\sqrt{3})^2$

$$\begin{aligned}(4+\sqrt{2})^2-2(1-\sqrt{2})^2&=(16+4\sqrt{2}+4\sqrt{2}+2)-2(1-\sqrt{2}-\sqrt{2}+2)\\&=(18+8\sqrt{2})-2(3-2\sqrt{2})\\&=18+8\sqrt{2}-6+4\sqrt{2}\\&=12+12\sqrt{2}\end{aligned}$$

Many people are confused by this simplification, but $8x + 4x = 12x$ whether x is rational or not…

Rationalising the denominator

It is preferable to have a rational denominator; therefore, if the denominator is irrational we must rationalise it.

Rationalise the denominators in the following quotients

By multiplying top *and* bottom by $\sqrt{3}$, we rationalise the denominator.

- $\dfrac{2}{\sqrt{3}}=\dfrac{2}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{2\sqrt{3}}{3}$
- $\dfrac{6}{\sqrt{2}}=\dfrac{6}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{6\sqrt{2}}{2}=3\sqrt{2}$

Surds

Simplify the following. Full workings are expected.

1. $\sqrt{28}$
2. $\sqrt{54}$
3. $\sqrt{32}-\sqrt{8}$
4. $\sqrt{11}+\sqrt{44}-\sqrt{99}$
5. $\sqrt{18}\times\sqrt{8}$
6. $\dfrac{21}{\sqrt{7}}$
7. $\dfrac{16}{\sqrt{32}}$
8. $\dfrac{3\sqrt{12}}{2\sqrt{24}}$
9. $\left(3\sqrt{2}\right)^3$
10. $\sqrt{12\frac{1}{4}}$

Completing the square

Complete the square for the following

1. x^2-8x+9
2. x^2+10x
3. x^2-3x+3
4. $2x^2-12x-3$
5. $5x^2-25x+9$
6. $3+2x-x^2$

Indices

Evaluate the following

1. 5^{-3}
2. $16^{1/4}$
3. $8^{-2/3}$
4. $\dfrac{10^8\times10^{-3}}{10^{-1}}$
5. $\left(4^{-5}\right)^{1/2}$
6. $3a^{-4}\times2a^{-2}$
7. $\dfrac{(9a)^2}{(3a)^4}$
8. $\dfrac{6x\times12x^5}{8x^{-3}}$
9. $\left(16x^{-4}\right)^{3/4}$
10. $\left(x\sqrt{x}\right)^4\div\sqrt{x}$

P.T.O for solutions

Indices

1. 1/125	2. 2	3. 1/4	4. 1 000 000
5. 1/32	6. $6a^{-6}$	7. a^{-2}	8. $9x^9$
9. $8x^{-3}$	10. $x^{11/2}$		

Surds

1. $2\sqrt{7}$	2. $3\sqrt{6}$	3. $2\sqrt{2}$	4. 0
5. 12	6. $3\sqrt{7}$	7. $2\sqrt{2}$	8. $(3\sqrt{2})/4$
9. $54\sqrt{2}$	10. 7 / 2		

Completing the square

1. $(x-4)^2-7$	2. $(x+5)^2-25$	3. $(x-1.5)^2+0.75$	4. $2(x-3)^2-21$
5. $5(x-2.5)^2-89/4$	6. $-(x-1)^2+4$		