C2 Notes (Edexcel)

Unit C2 – Core Mathematics

The examination will consist of one 1½ hour paper. It will contain about nine questions of varying length. The mark allocations per question which will be stated on the paper. All questions should be attempted.

Formulae which candidates are expected to know are given in the appendix to this unit and these will not appear in the booklet, *Mathematical Formulae including Statistical Formulae and Tables*, which will be provided for use with the paper. Questions will be set in SI units and other units in common usage.

Candidates are expected to have available a calculator with at least the following keys: $+$, $-$, $\times$, $\div$, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, $\ln x$, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory. Calculators with a facility for symbolic algebra, differentiation and/or integration are not permitted.

Prerequisites

A knowledge of the specification for C1, its preamble and its associated formulae, is assumed and may be tested.

SPECIFICATION	NOTES
1. Algebra and functions	
Simple algebraic division; use of the Factor Theorem and the Remainder Theorem.	Only division by $(x + a)$ or $(x - a)$ will be required. Candidates should know that if $f(x) = 0$ when $x = a$, then $(x - a)$ is a factor of $f(x)$. Candidates may be required to factorise cubic expressions such as $x^3 + 3x^2 - 4$ and $6x^3 + 11x^2 - x - 6$. Candidates should be familiar with the terms 'quotient' and 'remainder' and be able to determine the remainder when the polynomial $f(x)$ is divided by $(ax + b)$.
2. Coordinate geometry in the (x, y) plane	
Coordinate geometry of the circle using the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$ and including use of the following circle properties: (i) the angle in a semicircle is a right angle; (ii) the perpendicular from the centre to a chord bisects the chord; (iii) the perpendicularity of radius and tangent.	Candidates should be able to find the radius and the coordinates of the centre of the circle given the equation of the circle, and vice versa.

3. Sequences and series

The sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $\|r\| < 1$.	The general term and the sum to n terms are required. The proof of the sum formula should be known.
Binomial expansion of $(1+x)^n$ for positive integer n. The notations $n!$ and $\binom{n}{r}$.	Expansion of $(a + bx)^n$ may be required.

4. Trigonometry

The sine and cosine rules, and the area of a triangle in the form $\frac{1}{2}ab \sin C$.	
Radian measure, including use for arc length and area of sector.	Use of the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ for a circle.
Sine, cosine and tangent functions. Their graphs, symmetries and periodicity.	Knowledge of graphs of curves with equations such as $y = 3 \sin x$, $y = \sin(x + \pi/6)$, $y = \sin 2x$ is expected.
Knowledge and use of $\tan \theta = \dfrac{\sin\theta}{\cos\theta}$, and $\sin^2\theta + \cos^2\theta = 1$.	
Solution of simple trigonometric equations in a given interval.	Candidates should be able to solve equations such as $\sin(x - \pi/2) = \frac{3}{4}$ for $0 < x < 2\pi$, $\cos(x + 30°) = \frac{1}{2}$ for $-180° < x < 180°$, $\tan 2x = 1$ for $90° < x < 270°$, $6\cos^2 x° + \sin x° - 5 = 0$ for $0 \le x < 360$, $\sin^2\left(x + \dfrac{\pi}{6}\right) = \frac{1}{2}$ for $-\pi \le x < \pi$.

5. Exponentials and logarithms

$y = a^x$ and its graph.

Laws of logarithms	To include $\log_a xy = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$, $\log_a x^k = k \log_a x$, $\log_a \frac{1}{x} = -\log_a x$, $\log_a a = 1$.
The solution of equations of the form $a^x = b$.	Candidates may use the change of base formula.

6. Differentiation

Applications of differentiation to maxima and minima and stationary points, increasing and decreasing functions.	The notation $\mathrm{f}''(x)$ may be used for the second order derivative. To include applications to curve sketching. Maxima and minima problems may be set in the context of a practical problem.

7. Integration

Evaluation of definite integrals.

Interpretation of the definite integral as the area under a curve.	Candidates will be expected to be able to evaluate the area of a region bounded by a curve and given straight lines. E.g. find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$. $\int x \, \mathrm{d}y$ will not be required.
Approximation of area under a curve using the trapezium rule.	E.g. evaluate $\int_0^1 \sqrt{(2x+1)}\,\mathrm{d}x$ using the values of $\sqrt{(2x+1)}$ at x = 0, 0.25, 0.5, 0.75 and 1.

Appendix C2 – Formulae

This appendix lists formulae that candidates are expected to remember and that may not be included in formulae booklets.

Laws of logarithms

$$\log_a x + \log_a y \equiv \log_a (xy)$$

$$\log_a x - \log_a y \equiv \log_a \left(\frac{x}{y}\right)$$

$$k \log_a x \equiv \log_a (x^k)$$

Trigonometry

In the triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\text{area} = \tfrac{1}{2} ab \sin C$$

Area

area under a curve $= \int_a^b y \,\mathrm{d}x \quad (y \geq 0)$

Core Mathematics C2

Candidates sitting C2 may also require those formulae listed under Core Mathematics C1.

Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Binomial series

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^n\mathrm{C}_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times \ldots \times r}x^r + \ldots \quad (|x| < 1, n \in \mathbb{R})$$

Logarithms and exponentials

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Geometric series

$$u_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Numerical integration

The trapezium rule: $\displaystyle\int_a^b y\ \mathrm{d}x \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

Factor and Remainder Theorem

Simple algebraic division; use of the Factor Theorem and the Remainder Theorem. Only division by (x + a) or (x – a) will be required.
Students should know that if f(x) = 0 when x = a, then (x – a) is a factor of f(x).
Students may be required to factorise cubic expressions such as x^3+3x^2-4 *and* $6x^3+11x^2-x-6$.
Students should be familiar with the terms 'quotient' and 'remainder' and be able to determine the remainder when the polynomial f(x) is divided by (ax + b).

We need to be able to find *a*, *b* and *c* in the following:
$2x^3+x^2+5x-8=(x-1)(ax^2+bx+c)$.

We do this by comparing coefficients.
When we multiply out $(x-1)(ax^2+bx+c)$ we can see that the x^3 coefficient is *a*.
Hence $a=2$.

When we multiply out $(x-1)(2x^2+bx+c)$ we can see that the x^2 coefficient is $b-2$.
Hence $b-2=1$ and so $b=3$.

When we multiply out $(x-1)(2x^2+3x+c)$ we can see that the *x* coefficient is $c-3$.
Hence $c-3=5$ and so $c=8$.

Hence we see that $2x^3+x^2+5x-8=(x-1)(2x^2+3x+8)$ (note that the constant term in $(x-1)(2x^2+3x+8)$ is -8 which acts as a check that the above is correct).

If we used the notation $\mathrm{f}(x)\equiv 2x^3+x^2+5x-8$ we could then write $\mathrm{f}(x)\equiv(x-1)(2x^2+3x+8)$.
We say that $(x-1)$ is a factor of $\mathrm{f}(x)$.

It follows from this that $\mathrm{f}(1)=(1-1)(2\times1^2+3\times1+8)=0\times13=0$.

Factor Theorem

So we see that if $(x-1)$ is a factor of $\mathrm{f}(x)$ then $\mathrm{f}(1)=0$.
In fact the following is true:

$$\boxed{(x-a) \text{ is a factor of } \mathrm{f}(x) \text{ if and only if } \mathrm{f}(a)=0}$$

We can use this to find factors of polynomials.

If we have to find a factor of $\mathrm{f}(x)=px^3+qx^2+rx+s$ we are looking for a value *a* such that $\mathrm{f}(a)=0$. SHORT CUT : <u>The only values of *a* that we need to try are factors of *s*</u>.

We simply calculate $\mathrm{f}(1)$, $\mathrm{f}(2)$ etc. until we get a value of zero.

For example if we were asked to factorise $\mathrm{f}(x) = x^3 + 6x^2 + 11x + 6$ we could calculate $\mathrm{f}(-1) = -1 + 6 - 11 + 6 = 0$.

We deduce, from the Factor Theorem, that $(x+1)$ is a factor of $\mathrm{f}(x)$.

We then use the above method to write $\mathrm{f}(x) = (x+1)(x^2 + 5x + 6)$.

We then need to factorise the quadratic (if possible). In this case it does factorise to give $\mathrm{f}(x) = (x+1)(x+2)(x+3)$.

Example 1
Find a if $(x-2)$ is a factor of $\mathrm{f}(x) = x^3 + ax^2 + 5x + 6$.

$(x-2)$ is a factor so $\mathrm{f}(2) = 0$ (by Factor Theorem)

So $\mathrm{f}(2) = 8 + 4a + 10 + 6 = 4a + 24 = 0$, hence $a = -6$.

Example 2
Find a and b in $\mathrm{f}(x) \equiv x^3 + ax^2 + bx - 12$ given that $(x+2)$ and $(x-3)$ are factors.

$(x+2)$ is a factor so $\mathrm{f}(-2) = 0$. Hence $\mathrm{f}(-2) \equiv -8 + 4a - 2b - 12 = 0$.

From this we see that $4a - 2b = 20$ and so $2a - b = 10$ (1).

$(x-3)$ is a factor so $\mathrm{f}(3) = 0$. Hence $\mathrm{f}(3) \equiv 27 + 9a + 3b - 12 = 0$.

From this we see that $9a + 3b = -15$ and so $3a + b = -5$ (2).

Adding (1) and (2) gives $5a = 5$. So $a = 1$. Plugging this back into (1) gives $b = -8$.

Remainder Theorem

When the function $\mathrm{f}(x)$ is divided by $(x+a)$ the remainder is $\mathrm{f}(-a)$.

When the function $\mathrm{f}(x)$ is divided by $(ax+b)$ the remainder is $\mathrm{f}\left(-\dfrac{b}{a}\right)$.

NB: If $(ax+b)$ is a factor of $\mathrm{f}(x)$ then the remainder is zero, that is, from the above result, $\mathrm{f}\left(-\dfrac{b}{a}\right)=0$. We know this result already from the factor theorem.

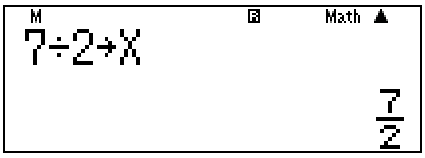
When we write $\mathrm{f}(x)$ in the form $\mathrm{f}(x)\equiv(x-a)g(x)+r$ we say that $\mathrm{g}(x)$ is the **quotient** and r is the **remainder**.

e.g. Find the remainder when $2x^3-3x^2-12x+2$ is divided by $2x-7$.

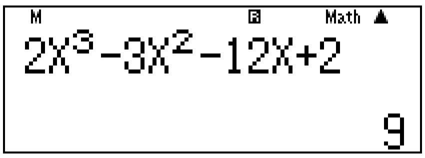
If we let $\mathrm{f}(x)=2x^3-3x^2-12x+2$ then the remainder we want is $\mathrm{f}\left(\dfrac{7}{2}\right)$.

We should proceed as follows:

Type in the following (on the Casio)



Then type in $2\mathrm{X}^3-3\mathrm{X}^2-12\mathrm{X}+2$ to get



So $\mathrm{f}\left(\dfrac{7}{2}\right)=2\left(\dfrac{7}{2}\right)^3-3\left(\dfrac{7}{2}\right)^2-12\left(\dfrac{7}{2}\right)+2=9$.

So the remainder when $2x^3-3x^2-12x+2$ is divided by $2x-7$ is 9.
We can also see that $2x^3-3x^2-12x+2=(2x-7)(x^2+2x+1)+9$

Example 3

(a) In the function $f(x) = x^3 + ax^2 + bx + 6$, a and b are integers. Find a and b given that:

(i) the remainder when $f(x)$ is divided by $(x-1)$ is 24.

(ii) the remainder when $f(x)$ is divided by $(x-2)$ is 60.

(b) Show that $(x+1)$ is a factor and factorise $f(x)$ completely.

The remainder theorem shows us that:

(i) can be rewritten as $f(1) = 24$.
(ii) can be rewritten as $f(2) = 60$.

$f(1) = 1^3 + a\times 1^2 + b\times 1 + 6 = 24$ and so $a + b = 17$ (1)
$f(2) = 2^3 + a\times 2^2 + b\times 2 + 6 = 60$ and so $4a + 2b = 46$. That is $2a + b = 23$ (2)

Solving (1) and (2) gives $a = 6$ and $b = 11$.

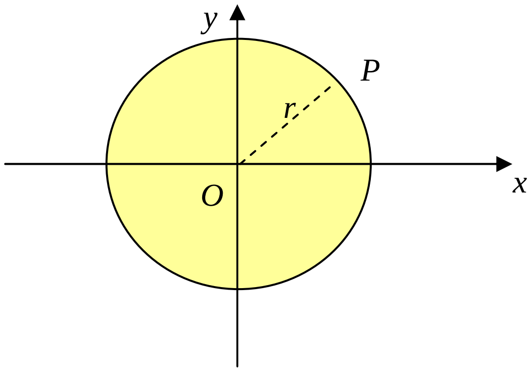
Hence $f(x) = x^3 + 6x^2 + 11x + 6$.

(b) $f(-1) = (-1)^3 + 6(-1)^2 + 11(-1) + 6 = 0$ so $(x+1)$ is a factor.

$$\begin{aligned} f(x) = x^3 + 6x^2 + 11x + 6 &= (x+1)(x^2 + 5x + 6) \\ &= (x+1)(x+2)(x+3) \end{aligned}$$

Coordinate geometry in the (x, y) plane

Coordinate geometry of the circle using the equation of a circle in the form $(x-a)^2+(y-b)^2=r^2$… Students should be able to find the radius and the coordinates of the centre of the circle given the equation of the circle, and vice versa.



Consider the circle shown above with radius r. If we look at the point P we see that, from Pythagoras $x^2+y^2=r^2$. This is the equation of a circle centre the origin, radius r.

If we moved the centre of the circle to (a, b) then the equation would be $(x-a)^2+(y-b)^2=r^2$.

Example 1
Find the centre and radius of the circle given by $x^2+y^2-4x+6y-3=0$

We need to complete the square on the x terms and the y terms.

$x^2+y^2-4x+6y-3=0$

$\Rightarrow (x-2)^2-4+(y+3)^2-9-3=0$

$\Rightarrow (x-2)^2+(y+3)^2=16$

So the centre is (2, -3) and the radius is 4.

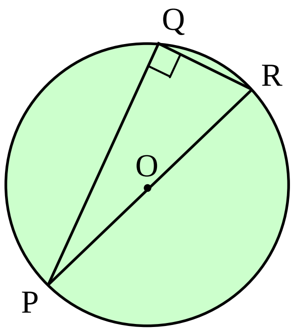
Example 2
The points A (3, 6) and B (9, 14) lie on a circle in such a way that AB is a diameter of the circle. Find the equation of the circle.

As AB is a diameter of the circle, the midpoint of AB must be the centre of the circle. Hence the centre of the circle is $\left(\frac{3+6}{2}, \frac{6+14}{2}\right)$, that is (6, 10).

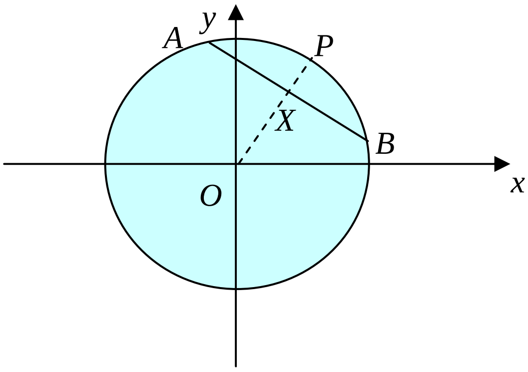
$AB=\sqrt{(6-3)^2+(14-6)^2}=10$ so the diameter is 10 and hence the radius is 5.

Hence the equation of the circle is $(x-6)^2+(y-10)^2=5^2=25$

…and including use of the following circle properties: (i) the angle in a semicircle is a right angle;

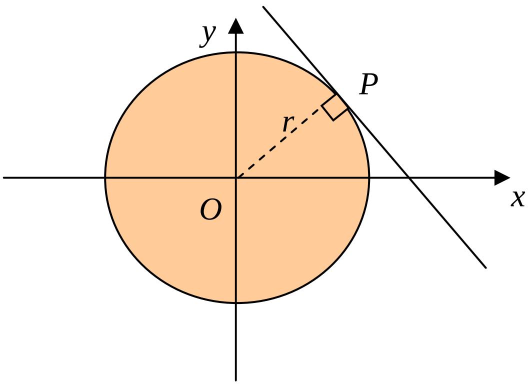


(ii) the perpendicular from the centre to a chord bisects the chord
If any chord is drawn on the circle then the perpendicular from the centre of the circle to the chord bisects the chord



This means that if we draw the chord AB on the circle and then draw the line through O perpendicular to AB, such that it hits AB at X. It follows that AX=XB.

(iii) the perpendicularity of radius and tangent.



In the diagram the tangent to the circle at P has been drawn. The line OP is at right angles to this tangent.

Geometric Series

The sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $|r| < 1$ *.Geometric series. The general term and the sum to n terms are required.*

We have seen that an *arithmetic sequence* is a sequence in which the difference between consecutive terms is constant. A *geometric sequence* is a sequence in which the *ratio* between consecutive terms is constant. (e.g. 3, 6, 12, 24, 48, …).

As before the first term is represented by *a* and the common ratio is represented by *r*.

In the example 3, 6, 12, 24, 48, … we could write it as $3, 3\times 2, 3\times 2^2, 3\times 2^3, 3\times 2^4 ....$.
In fact we could write it as $3\times 2^0, 3\times 2^1, 3\times 2^2, 3\times 2^3, 3\times 2^4 ....$

When we write it in this was we see that the *n*th term is $u_n = 3\times 2^{n-1}$.

In general if the first term is *a* and the common ratio is *r* then the first few terms of the sequence will be $a, ar, ar^2, ar^3 ....$ We can see from this that the *n*th term is $u_n = ar^{n-1}$.

Suppose that *S* is the sum of the first *n* terms of the sequence whose *n*th term is $u_n = ar^{n-1}$.

We can write $S_n = a + ar + ar^2 + ... + ar^{n-1}$

If we multiply both sides by *r* we get $rS_n = ar + ar^2 + ar^3 + ... + ar^n$

If we now subtract rS_n from *S* we get

$$\begin{aligned} S_n - rS_n &= \left(a + ar + ar^2 + ... + ar^{n-1}\right) - \left(ar + ar^2 + ar^3 + ... + ar^n\right) \\ &= a - ar^n \end{aligned}$$

Hence we see that $S_n(1-r) = a\left(1-r^n\right)$ and so $S_n = \dfrac{a\left(1-r^n\right)}{(1-r)}$.

The sum of the first *n* terms is $S_n = \dfrac{a\left(1-r^n\right)}{(1-r)}$. Alternatively we could have got $S_n = \dfrac{a\left(r^n-1\right)}{(r-1)}$

Example 1
The third and fifth terms of a geometric series are 40 and 10. Find the first term and the common ratio, given that it is positive.

Third term of 40 can be written as $ar^2 = 40$ (1)
Fifth term of 10 can be written as $ar^4 = 10$ (2)

Dividing (2) by (1) gives $r^2 = \frac{1}{4}$ and so, given that r is positive, $r = \frac{1}{2}$.

Plugging this back into (1) gives $a = 160$.

Example 2
Find the sum of the first 10 terms of the sequence :
(a) 5, 15, 45, 135,….

(b) 40, 20, 10, 5,….

(a) The first term, a, is 5 and the common ratio, r, is 3.
So, using the formula $S = \frac{a\left(1-r^n\right)}{(1-r)}$ we have $S = \frac{5\left(1-3^{10}\right)}{(1-3)} = 147620$.

(b) The first term, a, is 40 and the common ratio, r, is ½ .
So, using the formula $S = \frac{a\left(1-r^n\right)}{(1-r)}$ we have $S = \frac{40\left(1-\left(\frac{1}{2}\right)^{10}\right)}{\left(1-\frac{1}{2}\right)} = 80$ (to 2sf).

What would have happened if we had looked at the first 100 terms instead of the first 10 terms in the two examples we have just looked at?

In example 2(a) the value of S would have been very large.
In example 2(b) the value of S would have got much closer to 80.

The sum to infinity of a convergent geometric series.

In example 2(a) we could not have found the sum of an infinite number of terms since this would itself have been infinite. However in example 2(b) we could have found the sum of an infinite number of terms and the answer would have been 80.

What determines whether or not the sum of an infinite number of terms can be found or not? Look again at the formula $S = \dfrac{a\left(1-r^n\right)}{\left(1-r\right)}$.

The key part of this formula when we are considering an infinite number of terms is r^n.

If $r = 3$ (as in example 1) then r^n gets very large as n gets large.
If $r = \dfrac{1}{2}$ (as in example 2) then r^n tends towards zero as n gets large.

The rule is as follows..

If $-1 < r < 1$ then $r^n \to 0$ as $n \to \infty$, so the sum to infinity of the sequence is simply $S_\infty = \dfrac{a}{1-r}$.

If $r > 1$ or $r < -1$ then the sum to infinity of the sequence cannot be found.

Example 3

Find the infinite sum of 108, 36, 12, 4,….

The first term, a, is 108 and the common ratio, r, is $\frac{1}{3}$.

So, using the formula $S_\infty = \dfrac{a}{1-r}$ we have $S = \dfrac{108}{\left(1-\frac{1}{3}\right)} = 162$.

BLANK PAGE

Sequences and Series

Binomial expansion of $(1+x)^n$ *for a positive integer n. The notations* $n!$ *and* $\binom{n}{r}$. *Expansion of* $(a+bx)^n$ *may be required.*

If n is a positive integer then $n! = n\times(n-1)\times(n-2)\times....\times 2\times 1$.

In how many ways can we arrange the letters *p*, *q*, *r*, *s*, *t*?
The first could be any one of 5 letters, the second any one of 4 and so on. Hence we can arrange the letters *p*, *q*, *r*, *s*, *t* in $5!$ ways.

If we now replaced *p*, *q* and *s* with the letter *a* then this will reduce the number of arrangements by a factor of $3!$ (since *p*, *q* and *r* can be arranged in $3!$ ways).

If we then replaced *s* and *t* with the letter *b* then this will further reduce the number of arrangements by a factor of $2!$ (since *s* and *t* can be arranged in $2!$ ways).

Hence we see that if we the letters *a a a b b*, they can be arranged in $\frac{5!}{3!2!}$ ways.

In general if we had the letters

$$\underbrace{a\ a\ a\ a\ a\ a\ a\ a\ a\ a\ a}_{n-r}\ \underbrace{b\ b\ b\ b\ b\ b}_{r}$$

these could be arranged in $\frac{n!}{r!(n-r)!}$ ways. We use the notation $\binom{n}{r}$ or ${}^nC_r$ to represent $\frac{n!}{r!(n-r)!}$.

We can see from this that

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n, \qquad \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{1\times 2} \quad \text{and} \quad \binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{1\times 2\times 3}$$

and in general

$$\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)....(n-r+1)}{1\times 2\times 3\times ...\times r}}$$

NB : $0! = 1$ and so $\binom{n}{0} = \frac{n!}{0!n!} = 1$ and $\binom{n}{n} = \frac{n!}{n!0!} = 1$

Now consider the expansion of $(a+b)^5$.

We can write this as $(a+b)(a+b)(a+b)(a+b)(a+b)$.

When multiplying out we have to choose a letter from each of these 5 brackets.
We could choose *a* from 4 of the brackets and *b* from the other 1 bracket. Each of these will contribute a term of the form a^4b. Or we could choose *a* from 3 of the brackets and *b* from the other 2 brackets. Each of these will contribute a term of the form a^3b^2. And so on…

How many terms of the form a^3b^2 are there?

Each a^3b^2 term came from choosing *a* from 3 of the brackets and *b* from the other 2 brackets.

The number of a^3b^2 terms is, therefore, the same number as the number of arrangements of the letters *a a a b b*.

We saw that these could be arranged in $\frac{5!}{3!2!}$, or $\binom{5}{3}$, ways.

So the a^3b^2 term in the expansion of $(a+b)^5$ is $\binom{5}{3}a^3b^2$

Similarly the a^4b term in the expansion of $(a+b)^5$ is $\binom{5}{4}a^4b$.

So we see that $(a+b)^5 = \binom{5}{5}a^5 + \binom{5}{4}a^4b + \binom{5}{3}a^3b^2 + \binom{5}{2}a^2b^3 + \binom{5}{1}a^1b^4 + \binom{5}{0}b^5$.

In general we see that, as is stated on the C2 formulae sheet :

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots .b^n$$

and

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times\ldots\times r}x^r + \ldots ..x^n$$

Example 1

(a) Find the first three terms in the expansion of $(1+2x)^7$.

From the above we see that

$$(1+2x)^7 = 1+7(2x)+\frac{7\times 6}{1\times 2}(2x)^2+\frac{7\times 6\times 5}{1\times 2\times 3}(2x)^3+.....=1+14x+84x^2+280x^3+.....$$

(b) Hence find 1.002^7 to 6dp.

If we let $x=0.001$ then $(1+2x)^7=1.002^7$.

So from (a) we see that $1.002^7 \approx 1+0.014+0.000084+0.000000280$.

Hence see that $1.002^7 = 1.014084$ (to 6dp).

Example 2

Find the first three terms in the expansion of $(3-2x)^5$.

From the above we see that

$$\begin{aligned}(3-2x)^5 &= 3^5+5\times 3^4\times(-2x)+\frac{5\times 4}{1\times 2}\times 3^3\times(-2x)^2+\frac{5\times 4\times 3}{1\times 2\times 3}\times 3^2\times(-2x)^3+.....\\ &= 243-810x+9080x^2-720x^3\end{aligned}$$

Example 3

Find the x^3 term in the expansion of $(2+3x)(1+5x)^7$.

We can get an x^3 term by multiplying the 2 in $(2+3x)$ with the x^3 term in $(1+5x)^7$ or by multiplying the $3x$ in $(2+3x)$ with the x^2 term in $(1+5x)^7$.

The x^2 term in $(1+5x)^7$ is $\binom{7}{2}\times 1^5\times(5x)^2 = 21\times 5^2\times x^2 = 525x^2$.

The x^3 term in $(1+5x)^7$ is $\binom{7}{3}\times 1^4\times(5x)^3 = 35\times 5^3\times x^3 = 4375x^3$.

So $(2+3x)(1+5x)^7 = (2+3x)(...+525x^2+4375x^3+.....)$

Hence the x^3 term in the expansion of $(2+3x)(1+5x)^7$ is $3\times 525x^3+2\times 4375x^3 = 10325x^3$..

Example 4

If the first three terms of $\left(1+px\right)^5$ are 1, 20*x* and qx^2 then find *p* and *q*.

Using the same method as before we see that:

$$\left(1+px\right)^5 = {}^5C_0 + {}^5C_1\left(px\right) + {}^5C_2\left(px\right)^2$$
$$= 1 + 5px + 10p^2x^2$$

We are told that the first three terms are 1, 20*x* and qx^2.

Hence $5p = 20$ and so $p = 4$.

Also $10p^2 = q$ and so $q = 160$.

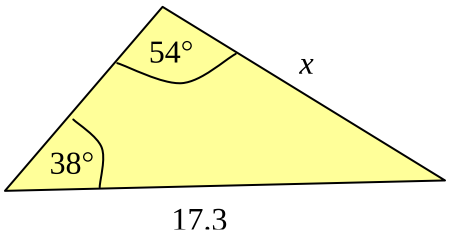
Trigonometry

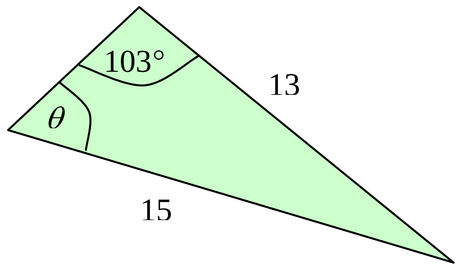
The sine and cosine rules, and the area of a triangle in the form $\frac{1}{2}ab\sin C$

Sine Rule

The Sine rule is written as $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ or $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

e.g. Find x and θ in the following:





$$\frac{x}{\sin 38} = \frac{17.3}{\sin 54}$$

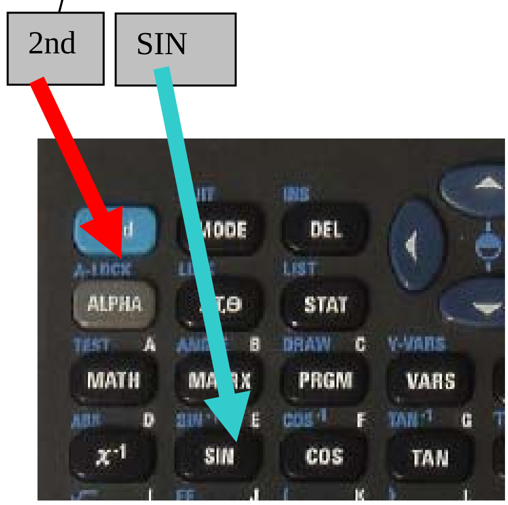
$$\Rightarrow x = \frac{17.3\sin 38}{\sin 54} = 13.2 \text{ cm (to3sf)}$$

$$\frac{\sin\theta}{13} = \frac{\sin 103}{15}$$

$$\sin\theta = \frac{13\sin 103}{15}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{13\sin 103}{15}\right) = 57.6° \text{ (to 1dp)}$$

You must use brackets on the calculator

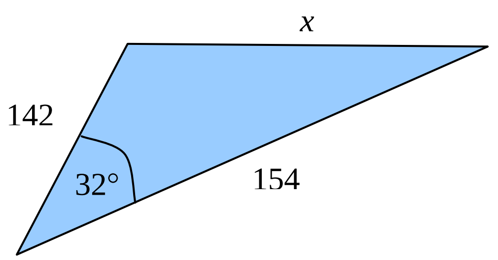


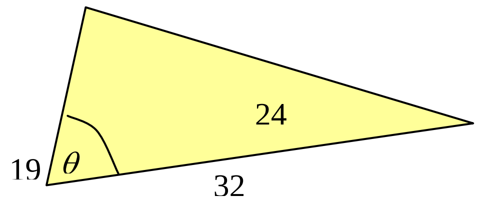
Cosine Rule

This is… $a^2 = b^2 + c^2 - 2bc\cos A$ and $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Example

Find x and θ in the following:

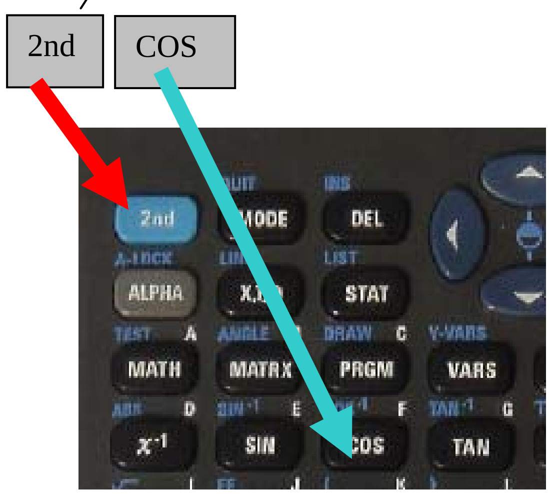




$$\begin{aligned} x^2 &= 142^2 + 154^2 - 2\times 142 \times 154\cos 32 \\ &= 6789 \\ \Rightarrow \quad x &= 82.4\,\text{cm}(\text{to } 3\text{sf}) \end{aligned}$$

$$\cos\theta = \frac{32^2 + 19^2 - 24^2}{2\times 32\times 19}$$

$$\Rightarrow \theta = \cos^{-1}(0.665\ldots) = 48.3^\circ\ (\text{to } 1\text{dp})$$



Use Cosine rule if you know either
(a) three sides or
(b) two sides and the enclosed angle.
Otherwise use Sine rule

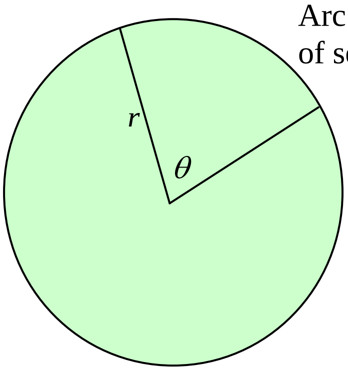
Area of triangle is $\frac{1}{2}ab\sin C$

Radian measure, including use for arc length and area of sector of a circle. Use of the formulae $s = r\theta$ *and* $A = \frac{1}{2}r^2\theta$ *for a circle.*

We are familiar with measuring one revolution in degrees and knowing it to be $360°$. Alternatively we could measure a revolution in radians, where one revolution is 2π radians.

So we see that to convert radians to degrees we multiply by $\frac{180}{\pi}$ and to convert degrees to radians we multiply by $\frac{\pi}{180}$.

It follows from the above that $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, $\frac{\pi}{3} = 60°$



Consider the sector shown above. The arc length is $\frac{\theta^\circ}{360}$ of the circumference of the circle, that is

$$\text{Arc length of sector} = \frac{\theta°}{360} \times 2\pi r = r\left(\theta° \times \frac{\pi}{180}\right).$$

The area of the sector is $\frac{\theta^\circ}{360}$ of the area of the circle, that is

$$\text{Area of sector } = \frac{\theta°}{360} \times \pi r^2 = \frac{1}{2}r^2\left(\theta° \times \frac{\pi}{180}\right)$$

It follows from the above that if we measure θ in radians then since $\theta° \times \frac{\pi}{180}$ is equal to θ radians we have:

Arc length of sector, $s = r\theta$
Area of sector, $A = \frac{1}{2}r^2\theta$

Exam Question

Figure 1

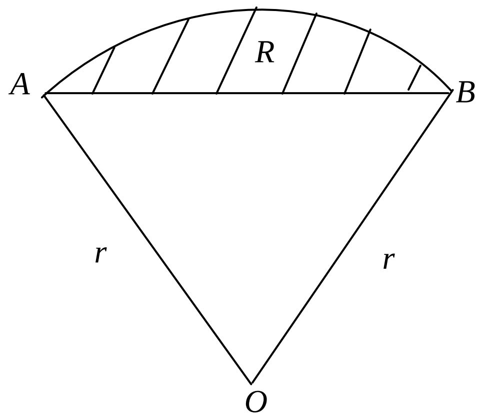


Figure 1 shows the sector OAB of a circle of radius r cm. The area of the sector is 15 cm^2 and $\angle AOB$ = 1.5 radians.

(a) Prove that $r = 2\sqrt{5}$.

(3)

(b) Find, in cm, the perimeter of the sector OAB.

(2)

The segment R, shaded in Fig 1, is enclosed by the arc AB and the straight line AB.

(c) Calculate, to 3 decimal places, the area of R.

(3)

Edexcel GCE Pure Mathematics P1 June 2003

(a) Area of sector $= \frac{1}{2}r^2\theta = 15$.

$\theta = 1.5$ so $\frac{1}{2}r^2 \times 1.5 = 15$. Hence we have $r^2 = 20$ and so $r = \sqrt{20} = 2\sqrt{5}$

(b) Arc length of sector $= r\theta = 2\sqrt{5} \times 1.5 = 3\sqrt{5}$

So total perimeter is $2\sqrt{5} + 2\sqrt{5} + 3\sqrt{5} = 7\sqrt{5}$

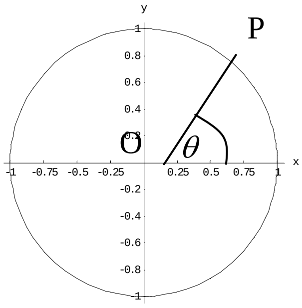
(c) Area of triangle AOB is $\frac{1}{2} \times 2\sqrt{5} \times 2\sqrt{5} \times \sin 1.5 = 10\sin 1.5 = 9.975$ cm^2 (to 3dp).

Area of R is $15 - 9.975 = 5.025$ cm^2 (to 3dp).

Trigonometry

Sine, cosine and tangent functions.

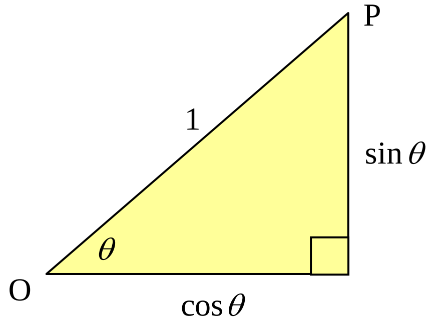
We need to define sin, cos and tan for angles outside of the range of $0°$ to $180°$ which we have used in triangles.



Consider the point P shown above which is 1 away from the origin O and is such that OP makes an angle of θ (measured anticlockwise) with the positive *x*-axis. $(\cos\theta, \sin\theta)$ are defined to be the coordinates of P and $\tan\theta$ is defined to be the gradient of OP.

Knowledge and use of $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ *and* $\sin^2\theta + \cos^2\theta = 1$.

Since $(\cos\theta, \sin\theta)$ are defined to be the coordinates of P we have the following triangle:



It follows from Pythagoras theorem on this triangle that $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

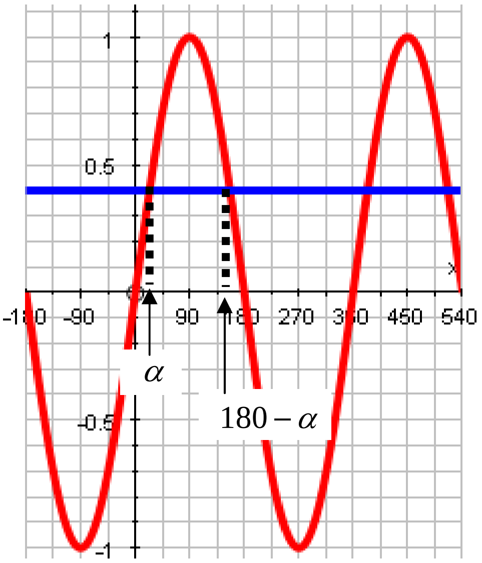
We use the notation $(\sin\theta)^2 = \sin^2\theta$ and $(\cos\theta)^2 = \cos^2\theta$.

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

It follows from our knowledge of gradients that, since $\tan\theta$ is the gradient of OP and since $(\cos\theta, \sin\theta)$ are the coordinates of P that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

Their graphs, symmetries and periodicity. Knowledge of graphs of curves with equations such as $y = 3\sin x$, $y = \sin\left(x + \frac{\pi}{6}\right)$, $y = 3\sin 2x$ *is expected.*

The graph of $y = \sin x$



Suppose we have to solve $\sin x = k$.

We draw $y = \sin x$ and $y = k$ (as above) and we look for the x-coordinates of the points of intersection.

It is clear from the graph that if α is the x-coordinate (from calculator) of one of the points of intersection then $180 - \alpha$ is the x-coordinate of another point of intersection.

The x-coordinates of all further points of intersection can be obtained from adding or subtracting multiples of 360 to either of these solutions.

Example

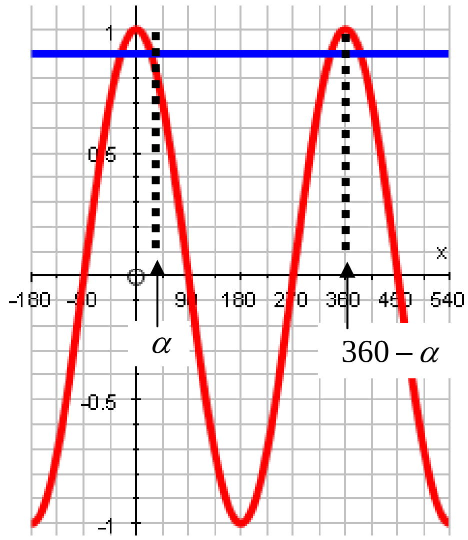
Solve $\sin x = 0.7$, $\quad 0^\circ \le x \le 360^\circ$

Our calculators give us one value, namely $\sin^{-1}(0.7) = 44.4^\circ$.

The other value, that appears on a different part of the curve is, from above, $180 - 44.4 = 135.6^\circ$ (to 1dp).

All other solutions are obtained by adding on or subtracting multiples of 360° to either of these answers.

The graph of $y = \cos x$



Suppose we have to solve $\cos x = k$.

We draw $y = \cos x$ and $y = k$ (as above) and we look for the x-coordinates of the points of intersection.

It is clear from the graph that if α is the x-coordinate (from calculator) of one of the points of intersection then $360 - \alpha$ is the x-coordinate of another point of intersection.

The x-coordinates of all further points of intersection can be obtained from adding or subtracting multiples of 360 to either of these solutions.

Example

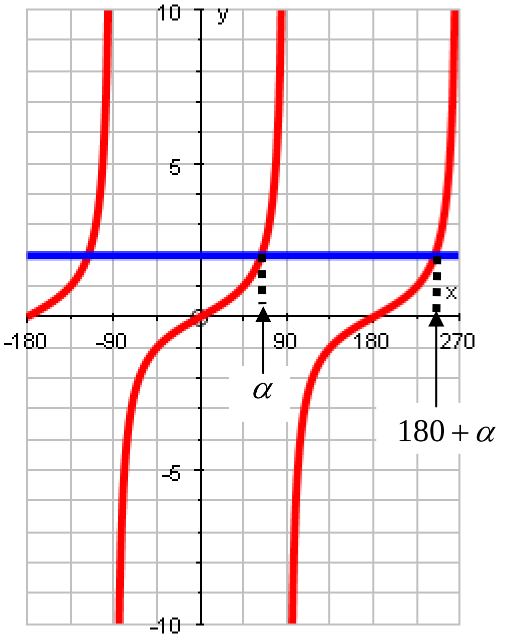
Solve $\cos x = -0.2$, $\quad 0^\circ \le x \le 360^\circ$

Our calculators give us one value, namely $\cos^{-1}(-0.2) = 101.5^\circ$

The other value, that appears on a different part of the curve is, from above, $360 - 101.5^\circ = 258.5^\circ$ (to 1dp).

All other solutions are obtained by adding on or subtracting multiples of 360° to either of these answers.

The graph of $y = \tan x$



Suppose we have to solve $\tan x = k$.
We draw $y = \tan x$ and $y = k$ (as above) and we look for the x-coordinates of the points of intersection.

It is clear from the graph that if α is the x-coordinates (from calculator) of one of the points of intersection then $180+\alpha$ is the x-coordinate of another point of intersection.

As before, all other solutions are obtained by adding on or subtracting multiples of $360°$ to either of these answers.
However we can see that when we do this it is the same as

Example
Solve $\tan x = 2$

Our calculators give us one value, namely $\tan^{-1}(2) = 63.4^\circ$. All other values are obtained by adding on multiples of $180°$ to this answer, to give 243.4°

In summary

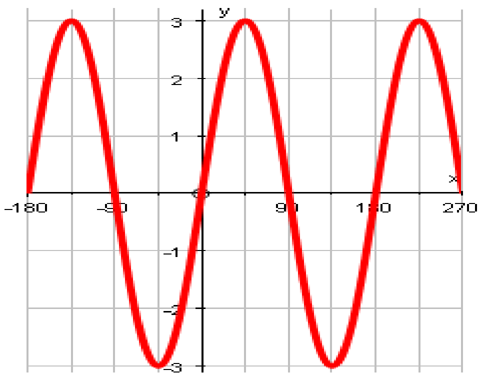
If α is a solution to $\sin x = k$ then so is $180 - \alpha$.

If α is a solution to $\cos x = k$ then so is $360 - \alpha$.

If α is a solution to $\tan x = k$ then so is $180 + \alpha$.

In <u>all three cases</u> all other solutions are found by adding or subtracting multiples of 360 to these two solutions.

From these graphs we see that $y = 3\sin 2x$, for example, is as follows:



The **period** of the curve is the value of *p* such that $f(x+p) = f(x)$ for all values of *x*.
So the period of $y = \sin x$ is $360°$ (or 2π in radians)
and the period of $y = \cos x$ is $360°$ (or 2π in radians)
and the period of $y = \tan x$ is $180°$ (or π in radians)

Solution of simple trigonometric equations in a given interval. Students should be able to solve equations such as

$$\sin\left(x - \frac{\pi}{2}\right) = \frac{3}{4} \quad \text{for} \quad 0 < x < 2\pi$$

$$\cos\left(x + 30°\right) = \frac{1}{2} \quad \text{for} \quad -180° < x < 180°$$

$$\tan 2x = 1 \quad \text{for} \quad 90° < x < 270°$$

$$6\cos^2 x° + \sin x° - 5 = 0 \quad \text{for} \quad 90° < x < 270°$$

$$\sin^2\left(x + \frac{\pi}{6}\right) = \frac{1}{2} \quad \text{for} \quad -\pi \le x < \pi$$

See example sheet for these.

A way of remembering sin, cos and tan for certain angles

	Angle, θ	0°	30°	45°	60°	90°
	Integers	0	1	2	3	4
	Square root	0	1	$\sqrt{2}$	$\sqrt{3}$	2
$\sin\theta$	Divide by 2	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	Flip order around	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	$\frac{\sin\theta}{\cos\theta}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Examples

1. Solve the following trigonometric equations in the given intervals :

(a) $\cos x = -\frac{4}{5}$ for $0^\circ \le x < 360^\circ$ (to 1dp)

$x = 143.1^\circ$ (from calc) or $360^\circ - (143.1^\circ) = 216.9^\circ$

$x = 143.1^\circ$ or 216.9°

(b) $\tan x = 3$ for $0^\circ \le x < 360^\circ$ (to 1dp)

$x = 71.6^\circ$ (from calc) or $180^\circ + 71.6^\circ = 251.6^\circ$

$x = 71.6^\circ$ or 251.6°

(c) $\sin x = -\frac{1}{2}$ for $0 \le x < 2\pi$ (exact values)

$x = -30^\circ$ (from calc) or $180^\circ - (-30^\circ) = 210^\circ$

We have to add 360° to -30° because -30° is not in range.

$x = 210^\circ$ or 330°

$x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

(d) $\sin x = \frac{2}{3}$ for $-180^\circ \le x < 180^\circ$ (to 1dp)

$x = 41.8^\circ$ (from calc) or $180^\circ - (41.8^\circ) = 138.2^\circ$

$x = 41.8^\circ$ or 138.2°

(e) $\cos x = \frac{1}{\sqrt{2}}$ for $0 \le x < 2\pi$ (exact values)

$x = 45^\circ$ (from calc) or $360^\circ - (45^\circ) = 315^\circ$

$x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

(f) $\sin x = \sqrt{3}\cos x$ for $-\pi \le x < \pi$ (to 2dp)

$\frac{\sin x}{\cos x} = \sqrt{3}\frac{\cos x}{\cos x}$

$\tan x = \sqrt{3}$

$x = 60^\circ$ (from calc) or $180^\circ + (60^\circ) = 240^\circ$

We have to subtract 360° from 240° because 240° is not in range.

$x = -120^\circ$ or 60°

$x = -\frac{2\pi}{3}$ or $\frac{\pi}{3}$

2. Solve the following trigonometric equations in the given intervals:

(a) $\cos\left(x+35^\circ\right)=-\frac{1}{2}$ for $-180^\circ < x \le 180^\circ$ (exact values)

$-145^\circ < x+35^\circ \le 215^\circ$

$x+35^\circ = 120^\circ$ (from calc) or $360^\circ - (120^\circ) = 240^\circ$

We have to subtract 360° from 240° because 240° is not in range.

$x+35^\circ = 120^\circ$ or -120°

$x = 85^\circ$ or -155°

(b) $\sin\left(x-\frac{\pi}{5}\right)=\frac{1}{2}$ for $0 < x \le 2\pi$ (exact values)

$-36^\circ < x-36^\circ \le 324^\circ$

$x-36^\circ = 30^\circ$ (from calc) or $180^\circ - (30^\circ) = 150^\circ$

$x = 66^\circ$ or 186°

$x = \frac{11\pi}{30}$ or $\frac{31\pi}{30}$

(c) $\tan\left(x-40^\circ\right)=2$ for $0^\circ < x \le 360^\circ$ (to 1dp)

$-40^\circ < x-40^\circ \le 320^\circ$

$x-40^\circ = 63.4^\circ$ (from calc) or $180^\circ + (63.4^\circ) = 243.4^\circ$

$x = 103.4^\circ$ or 283.4°

3. Solve the following trigonometric equations in the given intervals:

(a) $\tan 3x = 1$ for $0^\circ < x \le 180^\circ$

$0^\circ < 3x^\circ \le 540^\circ$

$3x = 45^\circ$ (from calc) or $180^\circ + (45^\circ) = 225^\circ$

We have to add 360° to get all values in range.

$3x = 45^\circ$ or 225° or 405°

$x = 15^\circ$ or 75° or 135°

(b) $\cos 4x = \dfrac{\sqrt{3}}{2}$ for $0° < x \le 180°$

$0^\circ < 4x^\circ \le 720^\circ$

$4x = 30^\circ$ (from calc) or $360^\circ - (30^\circ) = 330^\circ$

We have to add 360° to get all values in range.

$4x = 30^\circ$ or 330° or 390° or 690°

$x = 7.5^\circ$ or 82.5° or 97.5° or 172.5°

(c) $\sin 2x = -\dfrac{1}{2}$ for $0 < x \le \pi$

$0^\circ < 2x \le 360^\circ$

$2x = -30^\circ$ (from calc) or $180^\circ - (\text{-}30^\circ) = 210^\circ$

-30° is not in range so we have to add 360° to get 330°.

$2x = 210^\circ$ or 330°

$x = 105^\circ$ or 165°

$x = \dfrac{7\pi}{12}$ or $\dfrac{11\pi}{12}$

4. Solve the following trigonometric equations in the given intervals :

(a) $2\cos^2 x + \sin x - 1 = 0$ for $0 \le x < 2\pi$ (exact values)

Use the fact that $\cos^2 x + \sin^2 x = 1$

Replace $\cos^2 x$ with $1 - \sin^2 x$

$2\left(1 - \sin^2 x\right) + \sin x - 1 = 0$

Replace $\sin x$ with s

$2 - 2s^2 + s - 1 = 0$

$2s^2 - s - 1 = 0$

$(2s + 1)(s - 1) = 0$

$s = -\dfrac{1}{2}$ or $s = 1$

Solve $\sin x = -\dfrac{1}{2}$

$x = -30^\circ$ (from calc) or $180^\circ - (-30^\circ) = 210^\circ$

-30° is not in range so we have to add 360° to get 330°.

$x = 210^\circ$ or 330°

Solve $\sin x = 1$

$x = 90^\circ$ (from calc) or $180^\circ - (90^\circ) = 90^\circ$

So $x = 90^\circ$ or 210° or 330°

(b) $12\sin^2 x + \cos x - 6 = 0$ for $0^\circ \le x < 360^\circ$ (to 1dp)

Use the fact that $\cos^2 x + \sin^2 x = 1$

Replace $\sin^2 x$ with $1 - \cos^2 x$

$12\left(1 - \cos^2 x\right) + \cos x - 6 = 0$

Replace $\cos x$ with c

$12 - 12c^2 + c - 6 = 0$

$12c^2 - c - 6 = 0$

$\left(3c + 2\right)\left(4c - 3\right) = 0$

$c = -\frac{2}{3}$ or $c = \frac{3}{4}$

Solve $\cos x = -\frac{2}{3}$

$x = 131.8^\circ$ (from calc) or $360^\circ - (131.8^\circ) = 228.2^\circ$

Solve $\cos x = \frac{3}{4}$

$x = 41.4^\circ$ (from calc) or $360^\circ - (41.4^\circ) = 318.6^\circ$

So $x = 131.8^\circ$ or 228.2° or 41.4° or 318.6°

5. Solve the following trigonometric equations in the given intervals:

(a) $\sin^2\left(x+\frac{\pi}{5}\right)=\frac{3}{4}$ for $0 \le x < 2\pi$

$36^\circ \le x+36^\circ < 396^\circ$

Either (i) $\sin\left(x+36^\circ\right)=\sqrt{\frac{3}{4}}$ or (ii) $\sin\left(x+36^\circ\right)=-\sqrt{\frac{3}{4}}$

(i) $x+36^\circ = 60^\circ$ (from calc) or $180^\circ - (60^\circ) = 120^\circ$

(ii) $x+36^\circ = -60^\circ$ (from calc) or $180^\circ - (-60^\circ) = 240^\circ$

-60° is outside of range so we add 360° to get 300°

So $x+36^\circ = 60^\circ$ or 120° or 240° or 300°

$x = 24^\circ$ or 84° or 204° or 264°

$x=\frac{2\pi}{15}$ or $\frac{7\pi}{15}$ or $\frac{17\pi}{15}$ or $\frac{22\pi}{15}$

(b) $\cos^2\left(x-50^\circ\right)=\frac{1}{2}$ for $0^\circ \le x < 360^\circ$

$-50^\circ \le x-50^\circ < 310^\circ$

Either (i) $\cos\left(x-50^\circ\right)=\sqrt{\frac{1}{2}}$ or (ii) $\cos\left(x-50^\circ\right)=-\sqrt{\frac{1}{2}}$

(i) $x-50^\circ = 45^\circ$ or $360^\circ - (45^\circ) = 315^\circ$

315° is outside of range so we subtract 360° to get -45°

(ii) $x-50^\circ = 135^\circ$ or 225°

$x-50^\circ = -45^\circ$ or 45° or 135° or 225°

$x = 5^\circ$ or 95° or 185° or 275°

BLANK PAGE

Logarithms and Exponential Functions

$y = a^x$ and its graph.

We have already met equations of the form $2^n = 8$ and we have only been able to solve these equations by inspection where n is either an integer or a fraction.

Now consider the equation $3^n = 7$. The index, *n*, is the power to which 3 must be raised to give 7.

As present we do not have any notation for what we call *n*. There is no function that we know of which enables us to express *n* in terms of 3 and 7.

It is clearly useful for there to be such a function and this is where we use logarithms. Another word for this index, *n*, is logarithm.

We write $n = \log_3 7$, or (in words) *n* is the log (base 3) of 7.
NB : $\log_{10} x$ is more simply known as $\log x$.

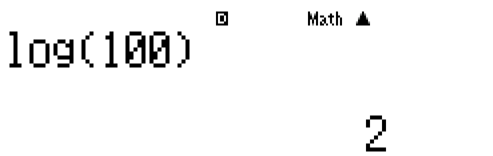
See how the *a* slides:

$$n = \log_a b \quad \Leftrightarrow \quad a^n = b$$

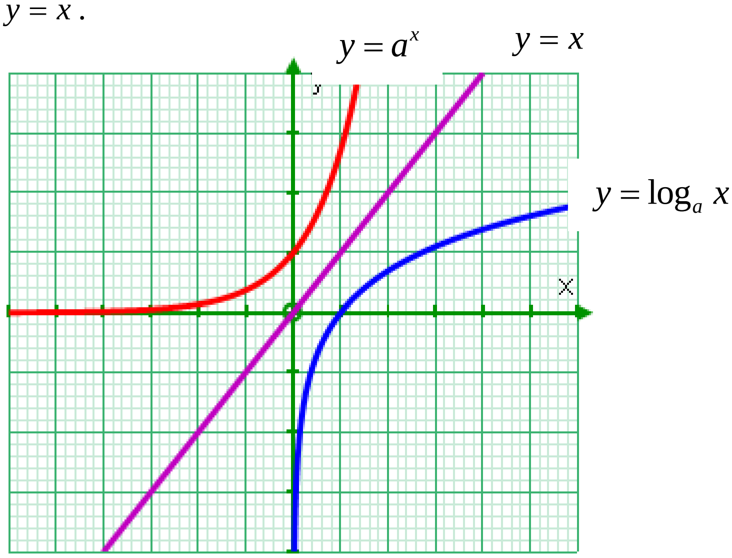
The key thing to remember about logarithms is this:

$$\boxed{a^n = b \quad \Leftrightarrow \quad n = \log_a b}$$

EXAM TIP:
If you forget this, type in log100 into your calculators. This gives us 2.
So we can write $2 = \log_{10} 100$ and we know that this can be rewritten as $10^2 = 100$.
Hence we can see that $n = \log_a b$ can be rewritten as $a^n = b$.



So, for example, if $x = \log_3 81$ then using the above we see that $3^x = 81$ and so $x = 4$.
We know that the graph of $y = a^x$ is an exponential graph and it follows that $y = \log_a x$ is a reflection of this in the line $y = x$.



NB $y = \log_a x$ does tend to infinity but it does so very slowly. For example $\log_{10} 1000000 = 6$.

Laws of logarithms. To include

$$\log_a xy = \log_a x + \log_a y$$
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$
$$\log_a x^k = k\log_a x$$
$$\log_a \frac{1}{x} = -\log_a x$$
$$\log_a a = 1$$

We have three laws of indices:

(I) $\quad a^x \times a^y = a^{x+y}$

(II) $\quad a^x \div a^y = a^{x-y}$

(III) $\quad \left(a^x\right)^y = a^{xy}$

If $a^x = m$ and $a^y = n$ then from the above result $x = \log_a m$ and $y = \log_a n$

From (I) we have that $mn = a^{x+y}$ and so $\log_a(mn) = x + y$

$$\text{Hence } \log_a(mn) = \log_a m + \log_a n$$

From (II) we have that $\dfrac{m}{n} = a^{x-y}$ and so $\log_a\left(\dfrac{m}{n}\right) = x - y$

$$\text{Hence } \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

From (III) we have that $m^y = \left(a^x\right)^y = a^{xy}$ and so $\log_a\left(m^y\right) = xy$ and so $y\log_a m = \log_a\left(m^y\right)$

$$\text{Hence } \log_a\left(m^y\right) = y\log_a m$$

Students may use the change of base formula.

The formula book has that $\log_a x = \dfrac{\log_b x}{\log_b a}$.

So, for example, $\log_a b = \dfrac{\log_{10} b}{\log_{10} a}$

The solution of equations of the form $b = a^x$.

To solve $b = a^x \ldots$
We saw earlier that $10^2 = 100$ can be rewritten as $2 = \log_{10} 100$.
In the same way $b = a^x$ can be rewritten as $x = \log_a b$.

Some calculators calculate $\log_a b$ directly, others do not.

If they don't, then use the above, that $\log_a b = \dfrac{\log_{10} b}{\log_{10} a}$.

So, in general:

The solution to $b = a^x$ is $x = \log_a b = \dfrac{\log_{10} b}{\log_{10} a}$.

Example 1
Solve $3^x = 7$.
We saw earlier that $10^2 = 100$ can be rewritten as $2 = \log_{10} 100$.
In the same way $3^x = 7$ can be rewritten as $x = \log_3 7$.

Some calculators calculate $x = \log_3 7$ directly to give 1.77 (to 3sf).

If they don't, then use the above, that $\log_3 7 = \dfrac{\log_{10} 7}{\log_{10} 3} = 1.77$ (to 3sf).

Example 2
If $k = \log_3 a$ then find the following in terms of k:

(a) $\log_3 a^2$

(b) $\log_3 27a$

(c) $\log_3 9a^3$

(a) $\log_3 a^2 = 2\log_3 a = 2k$

(b) $\log_3 27a = \log_3 27 + \log_3 a = 3 + k$

(c) $\log_3 9a^3 = \log_3 9 + \log_3 a^3 = \log_3 9 + 3\log_3 a = 2 + 3k$

Example 3
If $k = \log_3 x$ then find $\log_9 x$ in terms of x.

Using the above we see that $\log_9 x = \dfrac{\log_9 x}{\log_3 9} = \dfrac{k}{2}$

Example 4
Solve $\log_2(x+1) - \log_2 x = 3$.

Using first law of logarithm, we can rewrite the above as $\log_2\left(\dfrac{x+1}{x}\right) = 3$.

We saw earlier that $2 = \log_{10} 100$ can be rewritten as $10^2 = 100$.

In the same way $\log_2\left(\dfrac{x+1}{x}\right) = 3$ can be rewritten as $2^3 = \dfrac{x+1}{x}$.

Hence $8x = x + 1$.

So $7x = 1$ and hence $x = \dfrac{1}{7}$.

Example 5
Solve $\log_3(x+2) - \log_3 x = \log_3 4$.

Hence $\log_3(x+2) - \log_3 x - \log_3 4 = 0$.

Using first law of logarithm, we can rewrite the above as $\log_3\left(\dfrac{x+2}{4x}\right) = 0$.

We saw earlier that $2 = \log_{10} 100$ can be rewritten as $10^2 = 100$.

In the same way $\log_3\left(\dfrac{x+2}{4x}\right) = 0$ can be rewritten as $3^0 = \dfrac{x+2}{4x}$.

Hence $4x = x + 2$ and so $x = \dfrac{2}{3}$

Example 6
Solve $5^{2x} - 7\left(5^x\right) + 12 = 0$.

Putting $y = 5^x$ gives that $y^2 = 5^{2x}$.

Hence the above equation can be rewritten as $y^2 - 7y + 12 = 0$.

Factorising gives $(y-3)(y-4) = 0$.

Hence $y = 3$ or $y = 4$.

So $5^x = 3$ or $5^x = 4$

Using the method of earlier questions, we see that $x = 0.683$ (to 3sf) or $x = 0.861$ (to 3sf).

Example 7
Solve the following simultaneous equations given that *a* and *b* are positive.

$$a = 3b \qquad (1)$$
$$\log_3(a) + \log_3(b) = 2 \qquad (2).$$

(2) can be rewritten as $\log_3(ab) = 2$.

We saw earlier that $2 = \log_{10} 100$ can be rewritten as $10^2 = 100$.

In the same way $\log_3(ab) = 2$ can be rewritten as $ab = 3^2$, or simply $ab = 9$ (3).

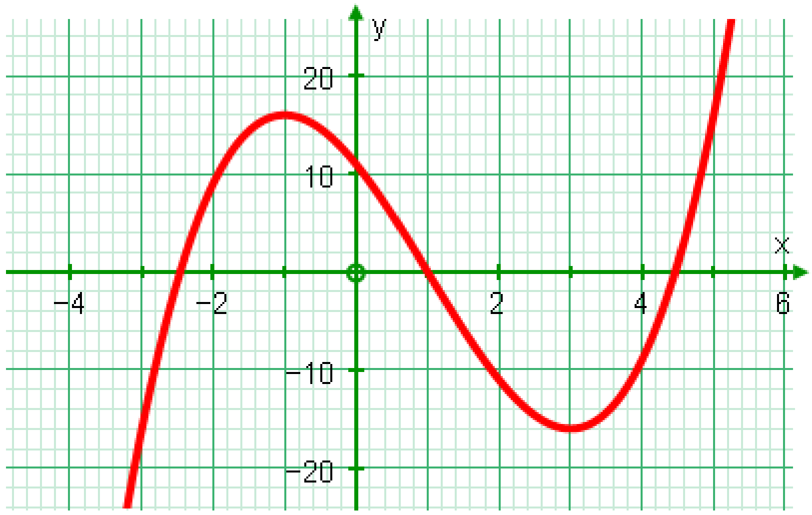
Substituting (1) into (3) gives $(3b)b = 9$ so $b^2 = 3$.

Hence $b = \sqrt{3}$ and so, from (1) $a = 3\sqrt{3}$

BLANK PAGE

Differentiation

Applications of differentiation to maxima and minima and stationary points, increasing and decreasing functions. The notation $f''(x)$ may be used for the second order derivative.
To include applications to curve sketching.
Maxima and minima problems may be set in the context of a practical problem.



Consider the curve $y = x^3 - 3x^2 - 9x + 11$ shown above.

The curve has positive gradient when $x < -1$ and when $x > 3$.

It has a negative gradient when $-1 < x < 3$.

It has zero gradient when $x = -1$ and when $x = 3$.

A curve is said to be increasing at any point if its gradient is positive at that point, i.e. $\frac{dy}{dx} > 0$.

A curve is said to be decreasing at any point if its gradient is negative at that point, i.e. $\frac{dy}{dx} < 0$.

A point on a curve is said to be a stationary point if $\frac{dy}{dx} = 0$ at that point.

So in the above example, the curve is increasing when $x < -1$ and when $x > 3$.

It is decreasing when $-1 < x < 3$.

It has stationary points at $x = -1$ and $x = 3$.

We can find the stationary points without the graph by calculating $\frac{\mathrm{d}y}{\mathrm{d}x}$ and solving $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$y = x^3 - 3x^2 - 9x + 11$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x - 9$.

Solving $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ gives us $3x^2 - 6x - 9 = 3\left(x^2 - 2x - 3\right) = 3\left(x - 3\right)\left(x + 1\right) = 0$

This gives $x = -1$ or $x = 3$.

The "gradient of the gradient" is denoted by $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

We see that in this example $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x - 6$

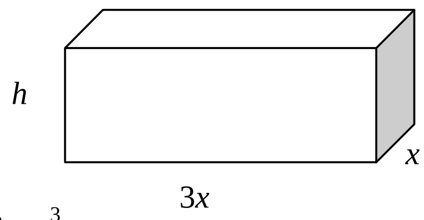
It is clear from the graph that the *y*-value at $x = -1$ is the largest value in that part of the curve. It is called a (local) maximum. We see that $x = -1$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -12 < 0$.

If $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is negative at a stationary point then the stationary point is a local maximum.

It is also true that if $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is positive at a stationary point then the stationary point is a local minimum.

Example 1

A factory produces cartons for ice cream. Each carton is in the shape of a closed cuboid with base dimensions $3x$ cm by x cm and height h cm.



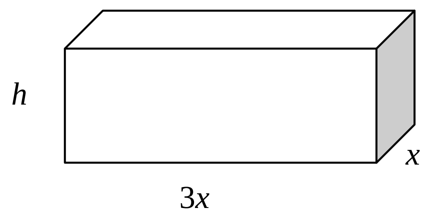
Given that the capacity of a carton has to be 1110 cm^3,
(a) express h in terms of x,
(b) find the surface area, A cm^2 in terms of x.
The factory wants to minimise the surface area of a carton.

(c) Find the value of x for which A is a minimum
(d) Prove that this value of A is a minimum.
(e) Calculate the minimum value of A.
.

(a) The volume of the cuboid is $3x \times x \times h = 3x^2 h$.

Since the volume is 1110, it follows that $3x^2 h = 1110$ and so $h = \dfrac{370}{x^2}$.

(b) The surface area of all six sides is $A = 3x^2 + 3x^2 + 3xh + 3xh + xh + xh = 6x^2 + 8xh$.



Using $h = \dfrac{370}{x^2}$ gives $A = 6x^2 + 8x\left(\dfrac{370}{x^2}\right) = 6x^2 + \dfrac{2960}{x}$.

(c) A is a minimum when $\dfrac{dA}{dx} = 0$.

$A = 6x^2 + 2960x^{-1}$

$\dfrac{dA}{dx} = 12x - 2960x^{-2} = 12x - \dfrac{2960}{x^2}$

If $\dfrac{dA}{dx} = 0$ then $12x = \dfrac{2960}{x^2}$.

Hence $12x^3 = 2960$ and so $x = 6.27$ cm (to 3sf).

(d) $\dfrac{d^2A}{dx^2} = 12 + 5920x^{-3} = 12 + \dfrac{5920}{x^3}$.

When $x = 6.27$, $\dfrac{d^2A}{dx^2} > 0$ so it is a minimum.

(e) When $x = 6.27$, $A = 708$ cm^3 (to 3sf)

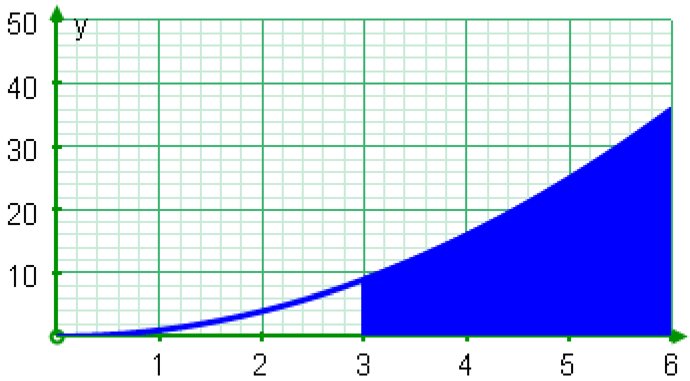
BLANK PAGE

Definite Integration

Evaluation of definite integrals. Interpretation of the definite integral as the area under a curve.

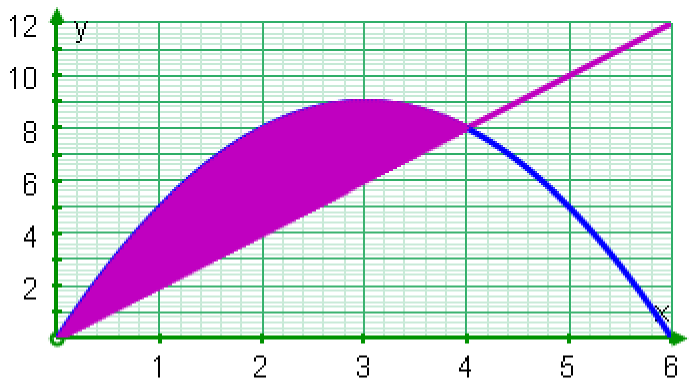
The area enclosed between the curve $y = x^2$, the x-axis, the lines $x = 3$ and $x = 6$ is denoted by
$\int_3^6 x^2 \, dx$.

This area is shaded in the diagram shown below.



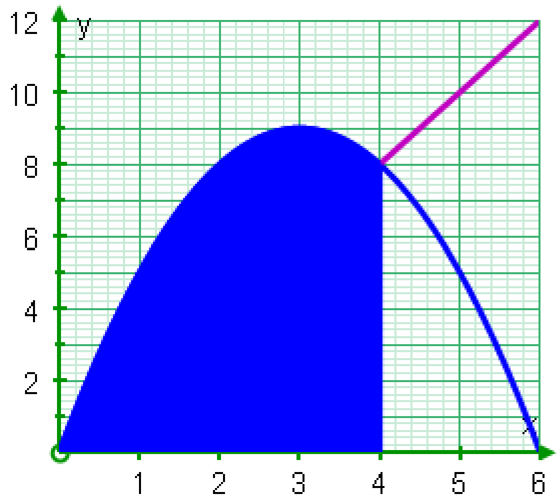
$$\int_3^6 x^2 \, dx = \left[\frac{1}{3}x^3\right]_3^6 = 72 - 9 = 63$$

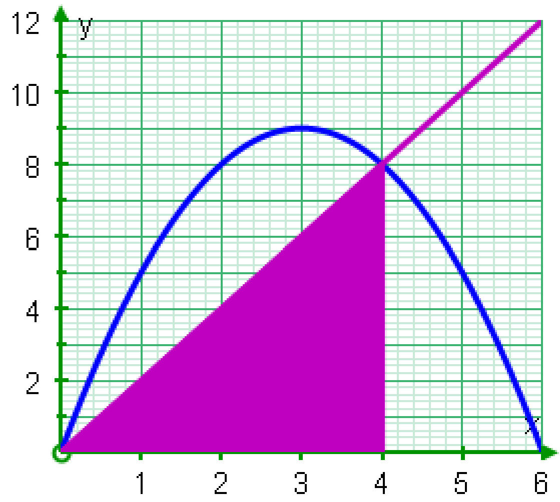
Students will be expected to be able to evaluate the area of a region bounded by a curve and given straight lines. E.g. find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$.
$\int x \, dy$ *will not be required.*



We need to find first of all the x-coordinates of the points of intersection of $y = 6x - x^2$ and $y = 2x$.
Solving these gives $x = 0$ and $x = 4$.

The area is then the difference between the two areas shown below:





So the area is $\int_0^4 6x - x^2 \, dx - \int_0^4 2x \, dx$.

This is equivalent to $\int_0^4 6x - x^2 - 2x \, dx = \int_0^4 4x - x^2 \, dx$.

Thus we see that the area is $\int_0^4 4x - x^2 \, dx = \left[2x^2 - \frac{1}{3}x^3 \right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$

So if the curves $y = f(x)$ and $y = g(x)$ intersect at $x = a$ and $x = b$ then the area enclosed between them is $\int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b f(x) - g(x) \, dx$

NB Use your calculators effectively on these.

Suppose we had to find $\int_1^{\frac{4}{3}} x^3 - 9x^2 + 24x - 16\mathrm{d}x = \left[\frac{x^4}{4} - 3x^3 + 12x^2 - 16x\right]_1^{\frac{4}{3}}$

Type in the following:

4	÷	3	SHIFT	RCL	)	OR
4	÷	3	STO $\rightarrow$	X, T		

You should now see the following display: "$4 \div 3 \rightarrow X$"

Then type in $\frac{x^4}{4} - 3x^3 + 12x^2 - 16x$ to get $\frac{-512}{81}$.

Then type in

1	SHIFT	RCL	)	OR
1	STO $\rightarrow$	X, T		

You should now see the following display: "$1 \rightarrow X$"

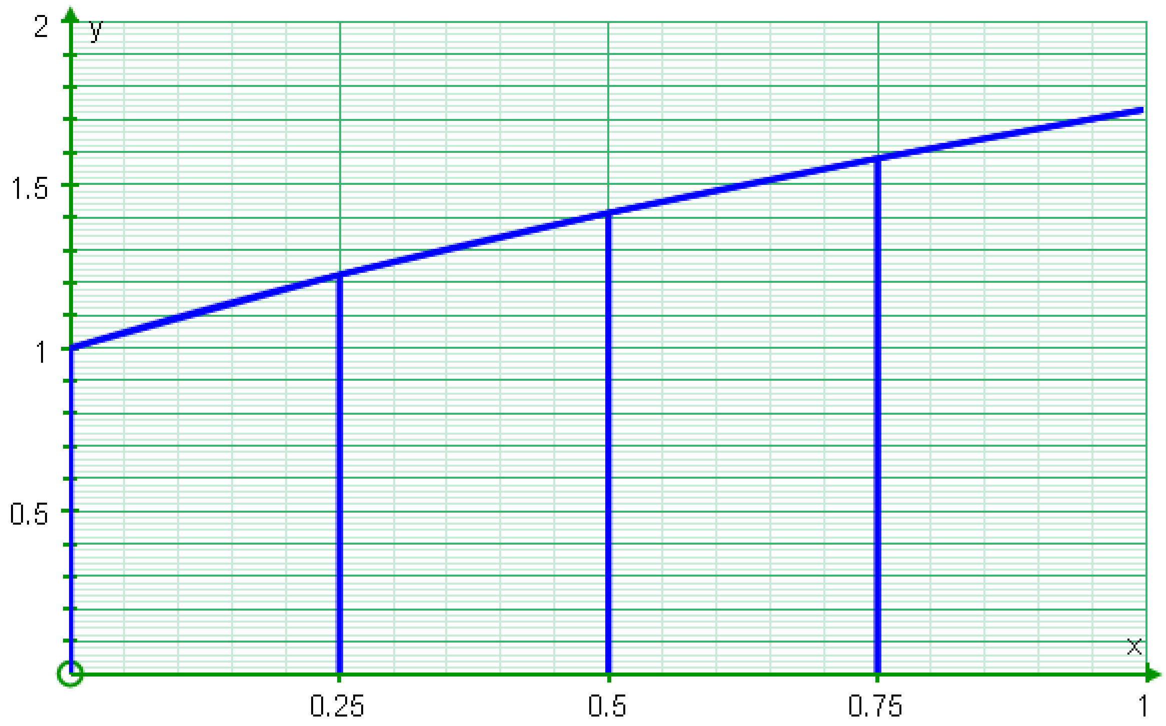
Then type in $\frac{x^4}{4} - 3x^3 + 12x^2 - 16x$ to get $\frac{-27}{4}$.

Hence

$$\int_1^{\frac{4}{3}} x^3 - 9x^2 + 24x - 16\mathrm{d}x = \left[\frac{x^4}{4} - 3x^3 + 12x^2 - 16x\right]_1^{\frac{4}{3}} = \frac{-512}{81} - \frac{-27}{4} = \frac{139}{324}$$

Approximation of area under a curve using the trapezium rule. E.g. evaluate $\int_0^1 \sqrt{2x+1}\,\mathrm{d}x$ *using the values of* x *at* x *= 0, 0.25, 0.5, 0.75 and 1.*

The area enclosed between the curve $y=\sqrt{2x+1}$, the x-axis, the lines $x=0$ and $x=1$ can be estimated using 4 trapezia.



We can calculate the y coordinates as follows:

x	0	0.25	0.5	0.75	1
$\sqrt{2x+1}$	1.00	1.22	1.41	1.58	1.73

The area of the 4 trapezia is therefore:

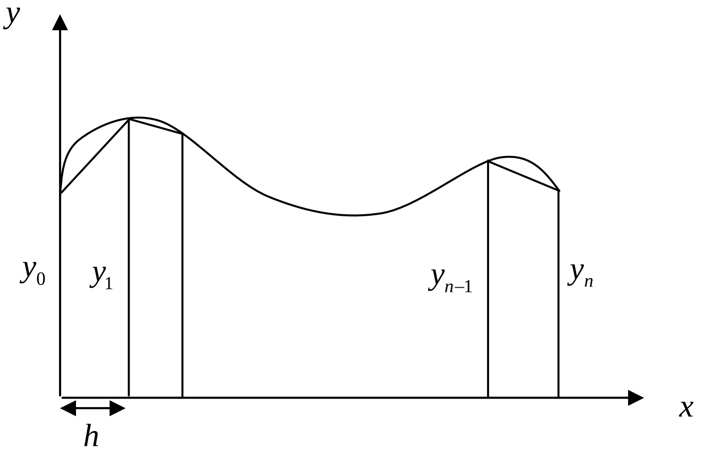
$$\frac{0.25}{2}(1.00+1.22)+\frac{0.25}{2}(1.22+1.41)+\frac{0.25}{2}(1.41+1.58)+\frac{0.25}{2}(1.58+1.73)$$

$$=\frac{0.25}{2}(1.00+1.22+1.22+1.41+1.41+1.58+1.58+1.73)$$

$$=\frac{0.25}{2}(1.00+2(1.22+1.41+1.58)+1.73)$$

$$=1.40$$

Consider the area under the curve shown below where n is the number of trapezia, h is the width of each trapezium and $y_0,\ y_1,\ ...y_{n-1},\ y_n$ are the heights as shown below:



The trapezium rule shows us that the area is approximately

$$\frac{h}{2}\left(y_0 + y_n + 2\left(y_1 + ... + y_{n-1}\right)\right)$$

As stated on the formula sheet,

$$\int_a^b y\ \mathrm{d}x \approx \frac{h}{2}\left(y_0 + y_n + 2\left(y_1 + ... + y_{n-1}\right)\right) \text{ where } h = \frac{b-a}{n}$$

Example 1

The table below shows the values of $y=\sqrt{1.6^x-1}$ for different values of x:

x	0	0.5	1	1.5	2	2.5	3
$y=\sqrt{1.6^x-1}$	0		0.775		1.249		1.760

(a) Fill in the blanks

(b) Use the trapezium rule to estimate $\int_0^3 \sqrt{1.6^x-1}\,\mathrm{d}x$.

(a) Use calculator to fill in the following:

x	0	0.5	1	1.5	2	2.5	3
$y=\sqrt{1.6^x-1}$	0	0.515	0.775	1.012	1.249	1.496	1.760

(b)

Formula book states that $\int_a^b y\,\mathrm{d}x \approx \frac{h}{2}\left(y_0+y_n+2\left(y_1+...+y_{n-1}\right)\right)$ where $h=\frac{b-a}{n}$.

How to use the formula….

a b

x	0	0.5	1	1.5	2	2.5	3
$y=\sqrt{1.6^x-1}$	0	0.515	0.775	1.012	1.249	1.496	1.760
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

So we see that $a=0$, $b=3$. The largest value of y_n is y_6 so $n=6$.

Hence $h=\frac{b-a}{n}=\frac{3-0}{6}=0.5$

More simply, h is the gap between the x values.

$$\int_0^3 \sqrt{1.6^x-1}\,\mathrm{d}x \approx \frac{0.5}{2}\left(0+1.760+2\left(0.515+0.775+1.012+1.249+1.496\right)\right)$$

So $\int_0^3 \sqrt{1.6^x-1}\,\mathrm{d}x \approx \frac{0.5}{2}\left(0+1.760+2\left(0.515+0.775+1.012+1.249+1.496\right)\right)=2.96$ (to 3sf).